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Discrete-Time Disturbance-Accommodating Control Theory with Applications to Missile Digital Control

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This paper describes an extension of the theory of (analog) disturbance-accommodating control to include a general class of control problems involving sampled-data and digital controllers. Digital control systems designed by the methods described herein will maintain performance specifications in the face of a wide range of disturbances encountered in practical applications. Design algorithms are derived for stabilization, set-point regulation, and servo-tracking digital control problems. Application of these results to the design of a digital control system for a missile is illustrated by an example.

I. Introduction: An Overview of the Disturbance/Control Problem

EFFECTIVE control of missiles and other dynamic systems requires the controller to be capable of dealing with a variety of uncertain "disturbances." Here, the term disturbances refers to external and internal stimuli that alter the systems behavior but that cannot be manipulated by the control designer. For instance, wind gusts, thrust misalignment, center of gravity (c.g.) offsets, drag asymmetry etc. are common examples of disturbances in missile control problems. Such disturbances often serve to disrupt or degrade the otherwise orderly behavior of the system being controlled. Thus, in addition to accomplishing the primary control task, a high-performance controller should have a reserve capacity for coping with such disruptions if and when they occur.

Modern control engineering has contributed a new approach to the design of controllers which can cope with uncertain disturbances. This new approach, introduced in Refs. 1-5, is called the theory of disturbance-accommodating control (DAC) and offers significant advantages over conventional design procedures.[6-8] The existing theory of DAC is restricted to *analog-type* control problems where sensor data is gathered and transmitted continuously and where the controller data processing is performed by conventional analog circuits. The purpose of this paper is to develop a general form of discrete-time, sampled-data DAC theory that can be applied to a wide variety of digital control system design applications. The proposed digital DAC design techniques are illustrated by an example concerned with the digital control of an interceptor missile.

II. Modes of Disturbance-Accommodation in DAC Theory

In the theory of disturbance-accommodating control there are three primary attitudes the control designer can take with respect to "accommodating" disturbances. Each of these attitudes leads to a different mode of control action. First, the designer can take the attitude that disturbances are totally unwanted and therefore the best control policy is to cancel out all effects caused by disturbances. This leads to a special form of DAC called a disturbance-absorbing controller. Second, the designer can take the alternative attitude that only certain parts or aspects of disturbances are unwanted. This leads to what is called a disturbance-minimization controller in which the controller cancels out only part of the disturbance effects. This latter form of DAC may be necessitated by physical constraints that prohibit total cancellation of disturbance effects.

The third possible attitude of disturbance accommodation is based on the fact that disturbances may sometimes produce effects which are *beneficial* to the primary control objectives. For instance, certain forms of wind gusts may actually "help" the controller to steer a missile toward a specified target. In such cases the best control policy is to let the disturbances assist the controller as much as possible. This attitude leads to what is called a disturbance-utilizing controller. In some applications designers may use a combination of these three primary modes of accommodation, thus leading to a multimode disturbance-accommodating controller.

Control engineering design procedures that accomplish the absorption, minimization, and utilization modes of disturbance accommodation just described have been developed for a broad class of traditional analog-type control problems. Those existing results are described in the tutorial references.[3,5] The purpose of the present paper is to develop a sampled-data discrete-time version of DAC theory which parallels the existing analog DAC theory in Ref. 5. Some additional topics concerning analog DAC theory are presented in Refs. 9-14, and a historical account of the subject may be found in Ref. 6.

III. Mathematical Models of Plants and Disturbances for Discrete-Time DAC

In order to develop a general purpose theory of DAC for digital control applications, it is necessary to derive appropriate mathematical models to represent the discrete-time behavior of plants and disturbances. A collection of such models is derived in this section.

A. Discrete-Time Models of Plants and Disturbances

The controlled and disturbed dynamical systems (plants) considered here are assumed to be modeled about appropriate operating points that lead to linearized differential equation models of the form

$$\dot{x} = A(t)x + B(t)u(t) + F(t)w(t) \qquad y = C(t)x \quad \text{(1a,b)}$$

where A, B, C, F are matrices, x is the plant state vector of n dimension, u is the plant control input vector of r dimension, w is the plant disturbance input vector of p dimension, and y is the plant output vector of m dimension. The objective of this section is to develop discrete-time models, corresponding to Eq. (1), which will describe the behavior of Eq. (1) at the isolated moments $t = t_0, t_0 + T, t_0 + 2T, \ldots,$ where T is a fixed positive constant (T = the "sampling period").

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It is recalled that the general solution of Eq. (1a) is given by Ref. 15:

$$x(t;t_0,u,w) = \Phi(t,t_0)x(t_0) + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)\mathrm{d}\tau + \int_{t_0}^{t} \Phi(t,\tau)F(\tau)w(\tau)\mathrm{d}\tau \tag{2}$$

where $\Phi(t,t_0)$ is the state-transition matrix for $A(t)$ in Eq. (1a). Setting $t \rightarrow (t_0+T)$ and then $t_0 \rightarrow t$ in Eq. (2) yields

$$x(t+T;t,u,w) = \Phi(t+T,t)x(t) + \int_{t}^{t+T} \Phi(t+T,\tau)B(\tau)u(\tau)\mathrm{d}\tau + \int_{t}^{t+T} \Phi(t+T,\tau)F(\tau)w(\tau)\mathrm{d}\tau \tag{3}$$

In sampled-data digital control systems the control action $u(t)$ is typically held constant between sampling times, $t_i \leq t < t_i + T$. Thus, if $u(t_i)$ denotes the constant value of u over such an interval, Eq. (3) may be written as

$$x(t_i+T;t_i,u,w) = \Phi(t_i+T,t_i)x(t_i) + \left[\int_{t_i}^{t_i+T} \Phi(t_i+T,\tau)B(\tau)\mathrm{d}\tau\right]u(t_i) + \int_{t_i}^{t+T} \Phi(t_i+T,\tau)F(\tau)w(\tau)\mathrm{d}\tau \tag{4}$$

Finally, if t_i is set equal to the "previous" sample time (t_0+nT), Eqs. (4) and (1b) can be written as†

$$x((n+1)T;nT,u,w) = \tilde{A}(nT)x(nT) + \tilde{B}(nT)u(nT) + \tilde{v}((n+1)T) \qquad y(nT) = C(nT)x(nT) \tag{5a}$$

where

$$\tilde{A}(nT) = \Phi(t_0+(n+1)T,t_0+nT)$$
$$\tilde{B}(nT) = \int_{t_0+nT}^{t_0+(n+1)T} \Phi(t_0+(n+1)T,\tau)B(\tau)\mathrm{d}\tau \tag{5b}$$

$$\tilde{v}((n+1)T) = \int_{t_0+nT}^{t_0+(n+1)T} \Phi(t_0+(n+1)T,\tau)F(\tau)w(\tau)\mathrm{d}\tau \tag{5c}$$

In the time-invariant case of Eq. (1), where A,B,C,F, are all *constant* matrices, the terms in Eqs. (5b) and (5c) simplify to

$$\tilde{A} = e^{AT} \qquad \tilde{B} = \int_0^T e^{A(T-\tau)}B\mathrm{d}\tau$$
$$\tilde{v} = \int_0^T e^{A(T-\tau)}Fw(\tau+t_0+nT)\,\mathrm{d}\tau \tag{6}$$

In accordance with standard procedures in DAC theory, the uncertain disturbances $w(t)$ in Eq. (1) are assumed to be modeled by a "waveform description" which permits $w(t)$ to be viewed as the "output" of a differential equation system[5]:

$$w(t) = H(t)z(t) \qquad \dot{z} = D(t)z + \sigma(t) \tag{7}$$

where $w = (w_1,...,w_p)$, $H(t)$, $D(t)$ are known, $z = (z_1,...,z_p)$ represents the state of the disturbance vector w, and $\sigma = (\sigma_1,...,\sigma_p)$ is a vector of impulse sequences $\sigma_i(t)$ which arrive in an unknown random-like manner with unknown intensities. It is assumed that the $\sigma_i(t)$ are "sparse"; that is the arrival times of the $\sigma(t)$ impulses are not spaced closer than some minimal spacing $\mu > 0$. Thus, $\sigma(t)$ is *not* white noise [see Ref. 5, pp. 421-422]. The specification of μ will be discussed later.

The incorporation of Eq. (7) into the discrete-time plant expression (5) is accomplished as follows. The general solution for $z(t)$ in Eq. (7) can be expressed as

$$z(t;t_0,\sigma) = \Phi_D(t,t_0)z(t_0) + \int_{t_0}^{t} \Phi_D(t,\xi)\sigma(\xi)\mathrm{d}\xi \tag{8}$$

where $\Phi_D(t,t_0)$ denotes the state-transition matrix for $D(t)$ in Eq. (7). Replacing t_0 by t_0+nT and $t=\tau$, $t_0+nT \leq \tau \leq t_0+(n+1)T$, in Eq. (8) permits the solution of Eq. (7) to be written as

$$w(\tau) = H(\tau)z(\tau)$$

$$z(\tau) = \Phi_D(\tau,t_0+nT)z(t_0+nT) + \int_{t_0+nT}^{\tau} \Phi_D(\tau,\xi)\sigma(\xi)\mathrm{d}\xi \tag{9}$$

Substituting Eq. (9) into Eq. (5c) yields

$$\tilde{v}((n+1)T) = \widetilde{FH}(nT)z(nT) + \tilde{\gamma}(nT) \tag{10a}$$

where

$$\widetilde{FH}(nT) = \int_{t_0+nT}^{t_0+(n+1)T} \Phi(t_0+(n+1)T,\tau) \times F(\tau)H(\tau)\Phi_D(\tau,t_0+nT)\mathrm{d}\tau \tag{10b}$$

$$\tilde{\gamma}(nT) = \int_{t_0+nT}^{t_0+(n+1)T} \left[\Phi(t_0+(n+1)T,\tau) \times F(\tau)H(\tau)\int_{t_0+nT}^{\tau} \Phi_D(\tau,\xi)\sigma(\xi)\mathrm{d}\xi\right]\mathrm{d}\tau$$

Consolidating Eq. (10) with Eq. (5) yields the discrete-time plant model as

$$x((n+1)T;nT,u,w) = \tilde{A}(nT)x(nT) + \tilde{B}(nT)u(nT) + \widetilde{FH}(nT)z(nT) + \tilde{\gamma}(nT) \tag{11a}$$

$$y(nT) = C(nT)x(nT) \tag{11b}$$

The reader is reminded that the arguments $(n+1)T$, nT in Eqs. (10a) and (11) actually denote the times $t=t_0+(n+1)T$, $t=t_0+nT$, respectively.

In the time-invariant case, $\tilde{A}$ and $\tilde{B}$ in the plant model [Eq. (11)] simplify as indicated in Eq. (6); $\widetilde{FH}$ and $\tilde{\gamma}$ simplify to

$$\widetilde{FH} = \int_0^T e^{A(T-\tau)}FHe^{D\tau}\mathrm{d}\tau$$

$$\tilde{\gamma} = \int_0^T e^{A(T-\tau)}FH\left[\int_0^{\tau} e^{D(\tau-\xi)}\sigma(\xi+t_0+nT)\mathrm{d}\xi\right]\mathrm{d}\tau \tag{12}$$

B. Composite Plant/Disturbance Model for Discrete-Time DAC

Setting $\tau = t_0+(n+1)T$ in Eq. (9), the discrete-time models [Eqs. (9) and (11)] corresponding to the continuous-time plant and disturbance equations (1), (7) can be expressed in

†It is important to note that the argument symbols $(n+1)T$ and nT in Eq. (5a) actually denote the time $t=t_0+(n+1)T$ and $t=t_0+nT$, respectively. This shorthand convention will be used consistently in this paper.

the composite block-matrix format

$$\left(\frac{x((n+1)T)}{z((n+1)T)}\right)=\left[\begin{array}{c|c}\tilde{A}(nT) & \widetilde{FH}(nT)\\ \hline O & \tilde{D}(nT)\end{array}\right]\left(\frac{x(nT)}{z(nT)}\right)$$

$$+\left[\frac{\tilde{B}(nT)}{O}\right]^{u(nT)}+\left(\frac{\tilde{\gamma}(nT)}{\tilde{\sigma}(nT)}\right) \tag{13a}$$

$$y(nT)=[C(nT)\,|O]\left(\frac{x(nT)}{z(nT)}\right) \tag{13b}$$

where

$$\tilde{D}(nT)=\Phi_D(t_0+(n+1)T,t_0+nT)$$

$$\tilde{\sigma}(nT)=\int_{t_0+nT}^{t_0+(n+1)T}\Phi_D(t_0+(n+1)T_1,\xi)\sigma(\xi)\mathrm{d}\xi \tag{14}$$

In the time-invariant case, $\tilde{D}$ and $\tilde{\sigma}$ in Eq. (14) simplify to

$$\tilde{D}=e^{DT}\qquad \tilde{\sigma}=\int_0^T e^{D(T-\xi)}\sigma(\xi+t_0+nT)\mathrm{d}\xi \tag{15}$$

The remainder of this paper is concerned with methods for designing $u(nT)$ in Eqs. (13) to achieve specified control objectives in the face of any admissible uncertain disturbance $w(nT)=H(nT)z(nT)$. For this purpose, $u(nT)$ will be sought in the extended state-feedback form $u(nT)=\phi(x(nT),z(nT),nT)$.

It should be noted that the terms $\tilde{\gamma}(nT)$ and $\tilde{\sigma}(nT)$ in Eqs. (13) represent the effects caused by the $\sigma(t)$ impulses which arrive during the interval *between* sampling instants, i.e., during $t_0+nT<t<t_0+(n+1)T$. Those effects, which will be called "residuals," result in perturbations in the values of $x((n+1)T)$ and $z((n+1)T)$, in accordance with Eqs. (13). Since the impulses in $\sigma(t)$ are assumed *completely unknown,* those perturbations due to residuals are likewise unknown, unpredictable, and unmeasurable. Therefore, to avoid excessive repetitions in errors associated with predictions of $x((n+1)T)$ using the discrete model (13), it will be necessary to assume that the $\sigma(t)$ impulses arrive in a once-in-a-while fashion with minimal spacing μ between impulses being somewhat *larger* than the sampling period T. As a practical matter it is advisable to have $\mu\geq 5T$.

C. Some Generalizations of Eqs. (13)

The discrete-time model (13) can be generalized to include various exceptional cases. For instance, the plant output $y(t)$ in Eq. (1) may be generalized to contain additional (linear) terms involving $u(t)$ and/or $w(t)$ in which case the discrete-time expression for $y(nT)$ in Eqs. (13) becomes

$$y(nT)=C(nT)x(nT)+P(nT)u(nT)+G(nT)w(nT)$$

$$w(nT)=H(nT)z(nT) \tag{16}$$

Cases of the form of Eq. (16) arise when, for instance, accelerometers are used to measure the "motions" of a missile.

Another generalization of Eqs. (13) consists of allowing plant-state dependent terms to appear in the disturbance model Eq. (7) in which case the discrete-time model for $w(nT)$, $z(nT)$ becomes

$$w(nT)=H(nT)z(nT)+L(nT)x(nT)$$

$$z((n+1)T)=\tilde{D}(nT)z(nT)+\tilde{M}(nT)x(nT)+\tilde{\sigma}(nT) \tag{17}$$

Situations requiring models of the type of Eq. (17) are referred to as cases of state-dependent disturbances.

IV. Composite State Observers for Discrete-Time DAC

A key element in the design of disturbance-accommodating controllers is the construction of a data processing device that can produce on-line, real-time estimates of both the plant state $x(t)$ and the disturbance state $z(t)$. Such a device is called a composite state observer, or state reconstructor, and general design recipes for the case of analog-type DAC's are given in Refs. 3, 5, and 11. In this section, design procedures are given for two forms of discrete-time composite state observers that represent natural extensions of the analog-type DAC observers derived in Ref. 11.

A. Full-Dimensional Composite State Observer for DAC

Recall from Eqs. (1) and (7) that x is n dimensional and z is ρ dimensional. The simplest form of a composite state observer for DAC is of dimension $(n+\rho)$ and, in the case of the discrete-time model Eqs. (13), has the form‡

$$\left(\frac{\mathrm{E}\hat{x}(nT)}{\mathrm{E}\hat{z}(nT)}\right)=\left[\begin{array}{c|c}\tilde{A}(nT) & \widetilde{FH}(nT)\\ \hline O & \tilde{D}(nT)\end{array}\right]\left(\frac{\hat{x}(nT)}{\hat{z}(nT)}\right)+\left[\frac{\tilde{B}(nT)}{O}\right]^{u(nT)}$$

$$+\left[\frac{K_{01}}{K_{02}}\right]\left[[C(nT)\,|O]\left(\frac{\hat{x}(nT)}{\hat{z}(nT)}\right)-y(nT)\right] \tag{18}$$

where $\hat{x}$, $\hat{z}$ denote the outputs produced by the observer and $u(nT)$, $y(nT)$ denote inputs to the observer that are on-line real-time sampled-data measurements of the plant control input and plant output. The two matrices K_{01}, K_{02} in Eq. (18) are "arbitrary" gain matrices that the designer must choose in accordance with certain desired response characteristics.

If the estimation errors are denoted by

$$\epsilon_x=x(nT)-\hat{x}(nT)\qquad \epsilon_z=z(nT)-\hat{z}(nT)$$

then it is easily shown that $\epsilon_x(nT)$, $\epsilon_z(nT)$ are governed by the coupled set of discrete-time equations

$$\left(\frac{\mathrm{E}\epsilon_x(nT)}{\mathrm{E}\epsilon_z(nT)}\right)=\left[\begin{array}{c|c}\tilde{A}+K_{01}C & \widetilde{FH}\\ \hline K_{02}C & \tilde{D}\end{array}\right]\left(\frac{\epsilon_x(nT)}{\epsilon_z(nT)}\right)+\left(\frac{\tilde{\gamma}(nT)}{\tilde{\sigma}(nT)}\right). \tag{19}$$

where for notational simplicity the argument (nT) has been omitted from the various matrices in Eq. (19). In order to produce reliable estimates $\hat{x}(nT)$, $\hat{z}(nT)$, the observer gain matrices K_{01}, K_{02} must be designed so that $\epsilon_x(nT)$, and $\epsilon_z(nT)$ in Eq. (19) both approach zero rapidly between impulses of $\sigma(t)$. This means that the solutions of the homogeneous part of Eq. (19) must be made strongly asymptotically stable to $\epsilon_x=\mathbf{0}$, $\epsilon_z=\mathbf{0}$. Since Eq. (19) is a *difference* equation, with possible time-varying coefficients, it is advisable in the most general case, to design K_{01}, K_{02} by means of the discrete Riccati equation method of optimal control theory. The details of that method may be found in Ref. 16.

If $\tilde{A}$, C, $\widetilde{FH}$, and $\tilde{D}$ in Eq. (19) are all *constant* matrices, the gain matrices K_{01}, K_{02} may be chosen as constant matrices so as to place the eigenvalues of the (constant) characteristic matrix

$$\mathfrak{A}=\left[\begin{array}{c|c}\tilde{A} & \widetilde{FH}\\ \hline O & \tilde{D}\end{array}\right]+\left[\frac{K_{01}}{K_{02}}\right]^{[C\,|\,O]}=\left[\begin{array}{c|c}\tilde{A}+K_{01}C & \widetilde{FH}\\ \hline K_{02}C & \tilde{D}\end{array}\right] \tag{20}$$

‡Hereafter, the shift operator $\mathrm{E}x(nT)$ means $x[(n+1)T]$.

at suitably damped locations within the unit circle of the complex plane. It is remarked that one attractive choice for the eigenvalue locations of $\mathcal{A}$ is to place all λ_i at zero. This choice produces "deadbeat" response of the homogeneous part of Eq. (19) which means that both $\epsilon_x(nT)$ and $\epsilon_z(nT)$ theoretically reach zero in a finite number of sample periods—assuming that the residuals $\tilde{\gamma}$, $\tilde{\sigma}$ are "quiet" during such an interval. This level of performance is not attainable from the conventional analog-type state observer. A block diagram of the composite observer Eq. (18) is shown in Fig. 1.

B. Reduced-Dimension Composite State Observer for DAC

The dimension of an observer indicates the number of delays (or integrators in analog systems) that the designer must use in the construction of the observer. Thus, there is an interest in seeking alternative observer designs that have a dimension less than $(n+\rho)$. One such observer, that has dimension $(n+\rho-m)$ where m, the dimension of $y(t)$, is assumed equal to the rank of C, can be constructed as follows. For notational simplicity, only the time-invariant case of Eqs. (13) will be considered here; the time-varying case is handled just as in Ref. 11.

First, let T_{12} and T_{22} be, respectively, any $n\times(n+\rho-m)$ and $\rho\times(n+\rho-m)$ matrices which satisfy the two conditions

$$[C \mid O]\left[\frac{T_{12}}{T_{22}}\right]=0 \qquad \text{rank}\left[\frac{T_{12}}{T_{22}}\right]=n+\rho-m \tag{21}$$

It is remarked that such matrices always exist, are nonunique, and are found easily. Next, define the two auxiliary matrices $\bar{T}_{12}$, $\bar{T}_{22}$ as

$$\bar{T}_{12}=(T_{12}^T T_{12}+T_{22}^T T_{22})^{-1}T_{12}^T$$

$$\bar{T}_{22}=(T_{12}^T T_{12}+T_{22}^T T_{22})^{-1}T_{22}^T$$

and set $C^{\#}=(CC^T)^{-1}C$. Then, a reduced-dimension composite state observer for the time-invariant case of Eq. (13) can be designed as follows. The first section of the observer consists of an $(n+\rho-m)$-dimension digital filter having the vector output variable $\xi(nT)$ and obeying the difference equation

$$\mathrm{E}\xi(nT)=(\tilde{\mathfrak{D}}+\Sigma\tilde{\mathfrak{K}})\xi(nT)+[(\bar{T}_{12}+\Sigma C)(\tilde{A}C^{\#T})$$
$$-(\tilde{\mathfrak{D}}+\Sigma\tilde{\mathfrak{K}})\Sigma]y(nT)+(\bar{T}_{12}+\Sigma C)\tilde{B}u(nT) \tag{22a}$$

where $y(nT)$, $u(nT)$ are filter inputs, Σ is an "arbitrary" gain matrix that the designer must choose to meet performance specifications, and

$$\tilde{\mathfrak{D}}=\bar{T}_{12}(\tilde{A}T_{12}+\widetilde{FH}T_{22})+\bar{T}_{22}\tilde{D}T_{22}$$

$$\tilde{\mathfrak{K}}=C(\tilde{A}T_{12}+\widetilde{FH}T_{22}) \tag{22b}$$

The second section of the observer consists of a pair of algebraic "assembly equations" which indicate just how the real-time filter output $\xi(nT)$ and the sampled-data plant output measurements $y(nT)$ are combined to form the estimates $\hat{x}(nT)$, $\hat{z}(nT)$. Those assembly equations are

$$\hat{x}(nT)=T_{12}\xi(nT)+[C^{\#T}-T_{12}\Sigma]y(nT) \tag{23a}$$

$$\hat{z}(nT)=T_{22}\xi(nT)-T_{22}\Sigma y(nT) \tag{23b}$$

The design of the gain matrix Σ is guided by the dynamic behavior of the estimation errors $(x-\hat{x})$ and $(z-\hat{z})$. It is straightforward to show that

$$(x-\hat{x})=T_{12}\epsilon(nT) \qquad (z-\hat{z})=T_{22}\epsilon(nT) \tag{24}$$

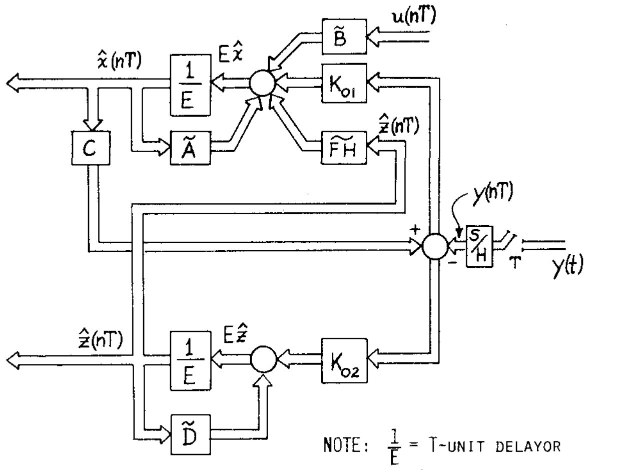

Fig. 1 Block diagram of full dimensional composite state observer [Eq. (18)] for discrete-time DAC.

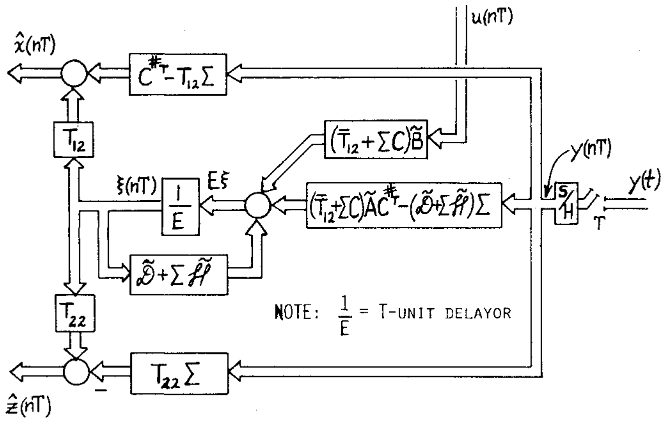

Fig. 2 Block diagram of reduced-dimension composite state observer [Eqs. (22) and (23)] for discrete-time DAC.

where the error variable $\epsilon(nT)$ is governed, between impulses of $\sigma(t)$, by the homogeneous difference equation

$$\mathrm{E}\epsilon(nT)=(\tilde{\mathfrak{D}}+\Sigma\tilde{\mathfrak{K}})\epsilon(nT) \tag{25}$$

Thus, in order to achieve $\hat{x}(nT)\approx x(nT)$, $\hat{z}(nT)\approx z(nT)$ promptly, the designer should choose Σ so that Eq. (25) is strongly asymptotically stable to $\epsilon=0$. This means that Σ should be chosen to place the eigenvalues of $(\tilde{\mathfrak{D}}+\Sigma\tilde{\mathfrak{K}})$ at appropriately damped locations within the interior of the unit circle in the complex plane. Therefore, all the remarks previously stated concerning the design of K_{01} and K_{02} in Eq. (20) apply also to the design of Σ in Eq. (25). In particular, the transpose of $(\tilde{\mathfrak{D}}+\Sigma\tilde{\mathfrak{K}})$ has the same form as Eq. (20).

A block diagram of the reduced-dimension observer Eqs. (22) and (23) is shown in Fig. 2.

V. Design of Discrete-Time DAC's for Digital Control Applications

In this section, design procedures are developed for synthesizing discrete-time disturbance-accommodating controllers to achieve disturbance absorption and disturbance minimization. (The disturbance-utilization mode of discrete-time DAC will be considered in a subsequent paper.) For notational simplicity only the time-invariant version of the general composite model Eqs. (13) is considered.

A. Design of Disturbance-Absorbing DAC Controllers for Discrete-Time Problems

The idea of disturbance absorption in discrete-time control problems is illustrated in Fig. 3. The disturbance $w(t)$ is

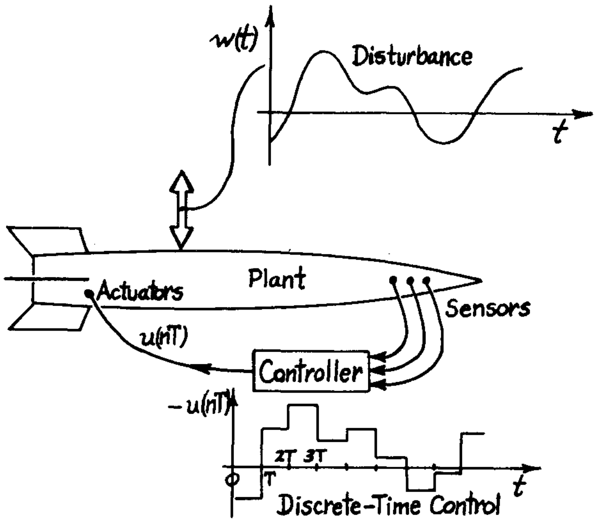

Fig. 3 Idea of disturbance absorption in discrete-time control problem.

generally a varying function of time that acts on the plant (for instance, a missile) continuously. On the other hand, the output of the digital controller is typically restricted to remain constant between sampling times. Thus in discrete-time disturbance-absorption control problems the designer must choose a constant value of $u(nT)$ which best counteracts the varying disturbance $w(t)$ over each sample interval $nT \leq t < (n+1)T$.

The approach to discrete-time disturbance absorption proposed in this paper is based on the structure of the discrete-time plant model Eq. (11a) which is repeated here

$$Ex(nT) = \tilde{A}x(nT) + \tilde{B}u(nT) + \widetilde{FH}z(nT) + \tilde{\gamma}(nT) \tag{26}$$

The term $\widetilde{FH}z(nT)$ in Eq. (26) will be called the "forecast" because it represents that part of $\Delta x = x[(n+1)T] - x(nT)$ which is forecasted to be contributed by the disturbance $w(t)$ acting during the interval $nT \leq t < (n+1)T$—assuming that the residual $\tilde{\gamma}(nT)$ is quiet during that interval. That forecast is based solely on the (presumed known) value of $z(nT)$ and the known dynamical model Eq. (7) of the disturbance behavior. Note that Eq. (26) does *not* indicate what influence $w(t)$ has on the behavior of $x(t)$ between the sample times $nT < t < (n+1)T$. Suppose $z(nT)$ could be measured directly. In that ideal case, if the constant value of control $u(nT)$ is chosen such that

$$\tilde{B}u(nT) = -\widetilde{FH}z(nT) \qquad (n = 0,1,2,\ldots,) \tag{27}$$

then the influence of $w(t)$ on the difference equation Eq. (26) will be completely cancelled, assuming $\tilde{\gamma}(nT) = 0$. This means that although $x(t)$ will still be perturbed by $w(t)$, those perturbations will momentarily go to zero at each of the sampling instants $t = t_0$, $t_0 + T$, $t_0 + 2T$, $t_0 + 3T$, etc. This latter effect is illustrated in Fig. 4.

The prescription Eq. (27) for discrete-time disturbance absorption is not physically realizable because $z(nT)$ cannot be measured directly. However, by using a composite state observer, as described in Sec. IV, the designer can closely approximate the ideal control Eq. (27) by choosing $u(nT)$ to satisfy

$$\tilde{B}u(nT) = -\widetilde{FH}\hat{z}(nT) \tag{28}$$

where the estimate $\hat{z}(nT)$ is obtained, in real time, from the observer.

The disturbance-induced "ripple" in $x(t)$ (Fig. 4) that will appear between sample times when the control strategy Eq. (27) is used, is referred to as "intersample ripple" and is

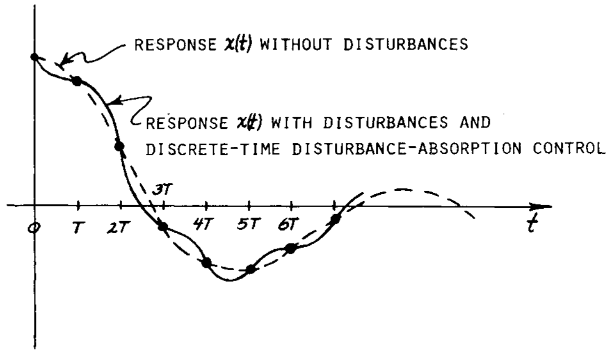

Fig. 4 Effect of discrete-time disturbance-absorption control policy [Eq. (27)] on motion of plant state $x(t)$ in Eq. (1).

inevitable in any digital control strategy where $u(nT)$ is constant and $w(t)$ is time varying. As part of the discrete DAC design procedure, the extent of that ripple should be investigated by simulation studies etc. to insure that it does not become unacceptably large. This consideration, in turn, becomes one of the factors which influences the choice of the controller sample period T. Namely, as T is made smaller, the disturbance-induced intersample ripple tends to diminish.

1. Allocation of Control Effort in DAC Design

In practical applications of disturbance-absorbing controllers, the control $u(\cdot)$ must simultaneously accomplish the primary control objective (such as stabilization, set-point regulation, or servotracking) *and* cancel out the disturbance effects. For this purpose, it is standard practice in DAC design procedure to allocate the total control effort into two task-oriented parts as follows:

$$u(t) = u_p(t) + u_d(t) \tag{29}$$

where u_d is responsible for accomplishing disturbance absorption (cancellation) and u_p is responsible for accomplishing the primary control objective. Thus, the disturbance-absorption prescription Eq. (28) would, in fact, be the design specification for the part $u_d(t)$ in Eq. (29).

2. Design Algorithm for $u_d(nT)$

The synthesis of a control $u_d(nT)$ which satisfies Eq. (28) can be accomplished as follows. First, the necessary and sufficient condition for existence of a control $u_d(nT)$ which satisfies Eq. (28) is given by the following theorem.

Existence condition for discrete-time disturbance-absorption control $u_d(nT)$:

A control $u_d(nT)$ that satisfies Eq. (28) for arbitrary $\hat{z}(nT)$ exists if, and only if,

$$\widetilde{FH} = \tilde{B}\tilde{\Gamma} \tag{30}$$

for some matrix $\tilde{\Gamma}$. ■

Condition Eq. (30) is called the "complete absorbability" condition for discrete-time DAC. It is remarked that condition Eq. (30) is the discrete-time version of the condition $FH = B\Gamma$ associated with continuous-time DAC theory (see Ref. 5, p. 439). In this regard, it should be noted that condition Eq. (30) may *fail* to be satisfied even though $FH = B\Gamma$ *is* satisfied. Thus, the property of "complete absorbability" may be lost as a consequence of converting from continuous-time to discrete-time control. When Eq. (30) does fail, the designer can pursue several alternative options as discussed in Sec. V.B.

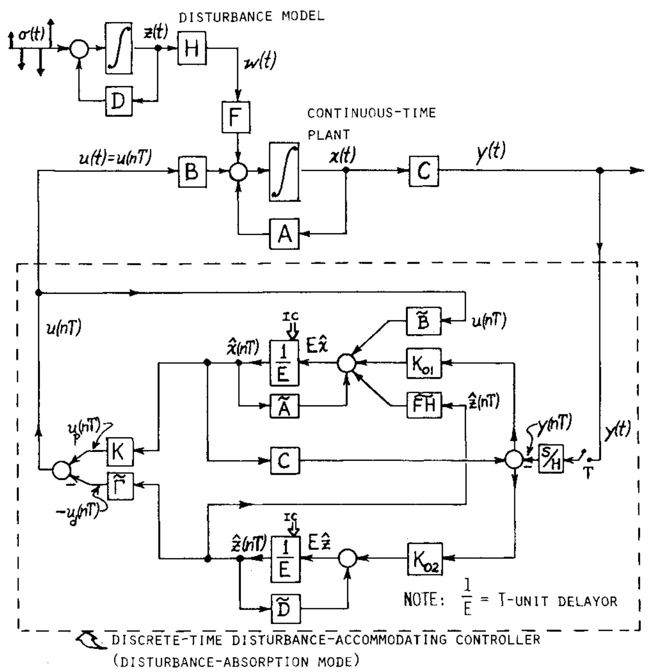

Fig. 5 General structure of discrete-time disturbance-absorbing controller.

Assuming condition Eq. (30) is satisfied, the part $u_d(nT)$ of the DAC control may be designed as

$$u_d(nT) = -\tilde{\Gamma}\hat{z}(nT) \tag{31}$$

The closed-loop dynamics of the discrete-time plant model Eq. (26), using Eqs. (29) and (31) and the definition $\epsilon_z = z - \hat{z}$ [see Eq. (19)], is governed by

$$\mathrm{E}x(nT) = \tilde{A}x(nT) + \tilde{B}u_p(nT) + \tilde{B}\tilde{\Gamma}\epsilon_z(nT) + \tilde{\gamma}(nT) \tag{32}$$

which shows that the effect of disturbances on Eq. (26) has been reduced to the residual $\tilde{\gamma}(nT)$ and the term $\tilde{B}\tilde{\Gamma}\epsilon_z(nT)$, where supposedly $\epsilon_z(nT) \to 0$ rapidly by virtue of the observer design. Since the $\sigma(t)$ impulses which produce the residual term $\tilde{\gamma}(nT)$ are assumed to arrive in a once-in-a-while fashion (sparse) and are completely unknown, unpredictable, and unmeasurable there is no scientific way to absorb the remaining disturbance effects in Eq. (32). In other words, the control Eq. (31) absorbs the disturbance effects in Eq. (26) as thoroughly as possible.

The primary control term $u_p(nT)$ in Eq. (32) can now be designed by well-known conventional techniques,[16] where the remaining disturbance-related terms $\tilde{B}\tilde{\Gamma}\epsilon_z(nT) + \tilde{\gamma}(nT)$ in Eq. (32) are ignored. The typical result is $u_p = K\hat{x}(nT)$ where $\hat{x}(nT)$ is generated by the composite state observer Eq. (18) or Eqs. (22) and (23).

A complete block diagram of the original continuous-time plant model Eq. (1), original continuous-time disturbance model Eq. (7), and the proposed disturbance-absorbing digital DAC controller Eqs. (29) and (31) [using the full-dimensional discrete-time composite state observer Eq. (18) and the typical structure of u_p] is shown in Fig. 5.

B. Design of Disturbance-Minimizing DAC Controllers for Discrete-Time Problems

The complete absorbability condition Eq. (30) must be satisfied in order to achieve total cancellation of the disturbance term $\widetilde{FH}z(nT)$ which appears in Eq. (26). In some cases it turns out that condition Eq. (30) cannot be satisfied owing to the structure of the given matrices A, B, F, H, D. For instance, when the disturbance $w(t)$ cannot be modeled satisfactorily as an essentially piecewise-constant function of time, no constant control $u(nT)$ can completely cancel all effects of $w(t)$ on $x((n+1)T)$, in general. (If rank $\tilde{B} = n$, such cancellation is possible; i.e., Eq. (30) is always satisfied.) In such cases the designer can take the alternative approach of designing $u_d(nT)$ in Eq. (29) to "best" approximate the ideal absorption condition Eq. (27), in some specified sense. Such DAC's are called disturbance-minimizing controllers.

There are literally hundreds of different ways to specify the "sense" in which a disturbance-minimizing controller is to approximate the ideal condition Eq. (27). In fact, once the designer realizes that Eq. (27) cannot be achieved owing to the failure of Eq. (30), it is possible that the best alternative approach will then not involve an attempt to approximate Eq. (27) but rather to take a whole new approach to the approximate disturbance-absorption strategy. These various issues are discussed at some length in Ref. 5 and will not be repeated here. In this section of the paper, the objective is to demonstrate the general steps one would take in designing a disturbance-minimizing controller for discrete-time applications.

1. The Norm Minimization Method of Disturbance-Minimization

The simplest approach to approximating the ideal disturbance-absorption condition Eq. (27) is to choose $u_d(nT)$ to minimize the vector "norm"

$$\min_{u_d} \| \tilde{B}u_d(nT) + \widetilde{FH}z(nT) \| \tag{33}$$

Condition Eq. (33) is a natural generalization of Eq. (27) because if condition Eq. (30) happens to be satisfied, the control $u_d(nT)$ which minimizes Eq. (33) will automatically satisfy Eq. (27); i.e., the norm Eq. (33) will then be made zero. The control $u_d(nT)$ which minimizes Eq. (33) is not unique, in general. However the control $u_d^0(nT)$ which minimizes Eq. (33) and which *itself* has minimum norm $\|u_d(nT)\|$ *is* unique and is given by

$$u_d^0(nT) = -\tilde{B}^\dagger \widetilde{FH}z(nT) \tag{34}$$

where $\tilde{B}^\dagger$ denotes the Moore-Penrose "generalized inverse" of $\tilde{B}$. If $\tilde{B}$ has rank equal to r, then $\tilde{B}^\dagger = (\tilde{B}^T\tilde{B})^{-1}\tilde{B}^T$. Otherwise $\tilde{B}^\dagger$ can be computed as described in Ref. 19. For physical realization, the control in Eq. (34) would be implemented as

$$u_d^0(nT) = -\tilde{B}^\dagger \widetilde{FH}\hat{z}(nT) \tag{35}$$

The remaining steps in the design and implementation of $u = u_p + u_d$ for disturbance minimization would be exactly the same as outlined in Sec. V.A for disturbance absorption.

2. The Critical State-Variable Method of Disturbance Minimization

Another possible approach to disturbance minimization is to attempt the "total" cancellation of disturbance effects on only certain specified "critical" state variables. In other words, try to realize the ideal condition Eq. (27) with respect to a specified subset of the plant state variables x_i rather than all of the x_i as indicated in Eq. (27). For this purpose the subset of critical state variables can be denoted by $C^*x = y^*$, for suitable choice of C^*, and the task of disturbance absorption is then to cancel the effect of $w(t)$ on the behavior of $y^*(nT)$. This can be accomplished by deriving a difference equation for $y^*(nT)$ which is forced by $\{u(nT), u[(n+1)T], \ldots, \text{etc.}, w(nT), w[(n+1)T], \ldots, \text{etc.}\}$, and is otherwise homogeneous in $y^*(nT)$, $y^*[(n+1)T]$, etc. Then, using Eq. (29) one can design u_d to absorb the disturbance terms in that difference equation and design u_p to properly control $y^*(nT)$. This technique is illustrated in Sec. VI of this paper (see also Ref. 5, pp. 462-468).

C. DAC Design for Discrete-Time Set-Point Regulator Problems

A typical set-point regulator problem in discrete time is stated as follows. Given the discrete-time plant/disturbance model Eq. (13), find a control $u(nT)$ which will steer $x(nT) \to x_{sp}$, and keep it there, in the face of any initial condition and any admissible disturbance $w(t)$. The set point x_{sp} is assumed given.

To design a DAC control $u(nT)$ for set-point regulation one can proceed as follows. Introduce the set-point error ϵ_{sp} defined as $\epsilon_{sp} = x_{sp} - x(nT)$ and note that the primary control objective is then to achieve $\epsilon_{sp}(nT) \to 0$, where $\epsilon_{sp}(nT)$ obeys the difference equation

$$\mathrm{E}\epsilon_{sp}(nT) = \tilde{A}\epsilon_{sp}(nT) - \tilde{B}u(nT) - \widetilde{FH}z(nT) - \tilde{A}x_{sp} - \tilde{\gamma}(nT) \tag{36}$$

Now, if u in Eq. (36) is split (allocated) in the usual manner [Eq. (29)], the task of u_d is to "accommodate" the combination of disturbance terms $-\widetilde{FH}z(nT) - \tilde{A}x_{sp}$ in Eq. (36). For instance, in the case of disturbance absorption, $u_d(nT)$ should be designed to ideally satisfy

$$\tilde{B}u_d(nT) = -\widetilde{FH}z(nT) - \tilde{A}x_{sp} \tag{37}$$

The necessary and sufficient condition for existence of a $u_d(nT)$ satisfying Eq. (37) is

$$[\widetilde{FH} \mid \tilde{A}x_{sp}] = \tilde{B}\tilde{\Gamma}_s \qquad \text{(for all admissible } x_{sp}) \tag{38a}$$

where $\tilde{\Gamma}_s$ is typically a function of x_{sp} having the form

$$\tilde{\Gamma}_s = [\tilde{\Gamma}_1 \mid \tilde{\Gamma}_2 x_{sp}] \qquad \tilde{B}\tilde{\Gamma}_1 = \widetilde{FH} \qquad \tilde{B}\tilde{\Gamma}_2 x_{sp} = \tilde{A}x_{sp} \tag{38b}$$

If the set point x_{sp} is an *arbitrary* vector, then it is necessary to further specify $\tilde{B}\tilde{\Gamma}_2 = \tilde{A}$ in Eq. (38b). Assuming Eqs. (38) are satisfied, the designer can synthesize $u_d(nT)$ (ideally) as

$$u_d(nT) = -\tilde{\Gamma}_1 z(nT) - \tilde{\Gamma}_2 x_{sp} \tag{39}$$

which will then leave Eq. (36) in the idealized form

$$\mathrm{E}\epsilon_{sp}(nT) = \tilde{A}\epsilon_{sp}(nT) - \tilde{B}u_p(nT) - \tilde{\gamma}(nT) \tag{40}$$

The remaining part $u_p(nT)$ of the control in Eq. (40) can now be designed in the form

$$u_p(nT) = -K\epsilon_{sp}(nT) \tag{41}$$

where K is chosen to achieve stabilization $\epsilon_{sp}(nT) \to 0$ in Eq. (40), assuming $\tilde{\gamma} = 0$. This implies the eigenvalues of $(\tilde{A} + \tilde{B}K)$ must lie interior to the unit circle. We therefore say a specific set point x_{sp} is "regulatable" if, and only if, $(\tilde{A}, \tilde{B})$ is stabilizable *and* Eqs. (38) are satisfied. Finally, the total DAC control $u = u_p + u_d$ in Eqs. (39) and (41) is physically realized as

$$u = u_p + u_d = -K[x_{sp} - \hat{x}(nT)] - \tilde{\Gamma}_1 \hat{z}(nT) - \tilde{\Gamma}_2 x_{sp} \tag{42}$$

The case of an output set-point $y_{desired} = y_{sp}$, $y_{sp} \in \mathcal{R}[C]$ (where $\mathcal{R}[\ ]$ denotes column range space) can be handled as follows. Since $y = Cx$, the condition $y(nT) \to y_{sp}$ implies that $x(nT) \to$ a value x^0 such that $Cx^0 = y_{sp}$. The set of all such x^0 is denoted by $X^0(y_{sp})$ and, assuming rank $C = m$, it is easy to show that

$$X^0 = \{x^0 \mid x^0 = C^T(CC^T)^{-1}y_{sp} + N\zeta\}$$

where $\mathcal{R}[N]$ forms a basis for the null space of C and ζ is an arbitrary $(n-m)$ vector. Thus, to achieve $y(nT) \to y_{sp}$ it is only necessary to regulate $x(nT)$ to any convenient "regulatable" state $x^0 \in X^0(y_{sp})$. This leads to a state set-point regulator problem of the type already discussed. It is remarked that the state x^0 having minimum norm corresponds to the choice $\zeta = 0$; however, there is no guarantee that such x^0 will be regulatable for any given y_{sp}.

D. DAC Design for Discrete-Time Servotracking Problems

The design of a discrete-time DAC to achieve servocontrol of a plant of the form Eq. (13) can be accomplished as follows. Assume, for instance, that the specified (given) servocommands $x_c(t)$ are modeled by the differential equation [compare with Eq. (7)]

$$\dot{x}_c = Rx_c + \mu_c(t) \tag{43}$$

where x_c is an n-vector and $\mu_c(t)$ is a sparse sequence of unknown impulses. The discrete-time model corresponding to Eq. (43) is

$$\mathrm{E}x_c(nT) = \tilde{R}x_c(nT) + \tilde{\mu}_c(nT) \qquad \tilde{R} = e^{RT} \text{ if } R = \text{const}$$

$$\tilde{\mu}_c = \int_{nT}^{(n+1)T} \Phi_R[(n+1)T, \xi]\mu_c(\xi)\mathrm{d}\xi \tag{44}$$

Now, introduce the servotracking error $\epsilon_s = x_c(nT)x(nT)$ and note that the primary control task is to achieve $\epsilon_s(nT) \to 0$, and keep it there, where $\epsilon_s(nT)$ obeys the difference equation

$$\mathrm{E}\epsilon_s(nT) = \tilde{A}\epsilon_s(nT) - \tilde{B}u(nT) - \widetilde{FH}z(nT) + (\tilde{R} - \tilde{A})x_c - \tilde{\gamma}(nT) + \tilde{\mu}_c(nT) \tag{45}$$

Setting $u = u_p + u_d$ as before, the task of u_d is to accommodate the collection of disturbance terms $-\widetilde{FH}z(nT) + (\tilde{R} - \tilde{A})x_c$ in Eq. (45), and the task of u_p is to steer $\epsilon_s(nT) \to 0$. For instance, in the case of disturbance absorption, u_d should be designed to achieve (ideally)

$$\tilde{B}u_d = -\widetilde{FH}z(nT) + (\tilde{R} - \tilde{A})x_c(nT)$$

where $x_c(nT)$ is assumed given in real time. From this point on, the synthesis of u_d and u_p proceeds exactly as in Eqs. (37-42). Note that the servocommand model Eq. (43) can be used to model the uncertain motions (or evasive maneuvers) of a target in missile-intercept problems. The case of output servotracking $y(nT) \to y_c(nT)$, where $y_c(nT)$ is given, can be handled by writing $y_c = Cx_c$ and proceeding as in the case of output set points where x_c plays the role of x^0. In that case, an additional observer is required to generate the "command state" x_c from on-line measurements of the command $y_c(nT)$. For that purpose, either the full-dimensional observer Eq. (18) or reduced-dimension observer Eqs. (22) and (23) may be used; set $z = 0$, $\rho = 0$, $T_{22} = 0$, $x = x_c$, etc. (see Ref. 4).

VI. Application of Discrete-Time DAC Theory to a Missile Digital Control Design

To demonstrate application of the discrete-time DAC theory to a missile-related digital control problem, we will consider the classical "planar-motion homing intercept" example, commonly referred to as the "small LOS angle" missile homing problem[17,18] for a point-mass missile. In the case considered here, the target is assumed *fixed,* and the relative motion between the missile and target along the LOS is assumed to occur at essentially a *known* constant velocity (closing velocity = known constant). Moreover, the significant control and disturbance forces acting on the missile are assumed to act normal to the LOS, and the corresponding motion of the missile normal to the reference LOS is assumed small enough to justify the assumption that the LOS angle is essentially constant during flight (i.e., LOS ≈ reference LOS).

Under the preceding assumptions, the displacement x_N of the missile *normal* to the reference LOS is governed, approximately, by the equation

$$\frac{d^2x_N}{dt^2}=u(t)+w(t) \qquad x_N(t_0)=x_{N,0} \qquad \dot{x}_N(t_0)=\dot{x}_{N,0} \tag{46}$$

where $u(t)$ denotes the control force and $w(t)$ denotes the collective disturbance effects; e.g., $w(t)$ represents the combined effects of all *un*controlled forces acting on the missile, normal to the LOS; see Fig. 6a. For instance $w(t)$ includes the effects of possible aerodynamic side loadings, lateral winds, the component of gravity normal to the LOS, etc. It is assumed that $w(t)$ cannot be measured directly and cannot be predicted accurately.

The primary control objective is to achieve $x_N(T_f)\approx 0$ at a prescribed intercept time $t=T_f$. It is clear from Eq. (46) that the uncertain disturbances $w(t)$ will typically interfere with achievement of the primary control objective. Thus, the control designer might consider the accommodation of $w(t)$ by using the disturbance-absorption mode of DAC. For that purpose, the total control effort u in Eq. (46) is split (allocated) a priori into two task-oriented parts $u=u_p+u_d$, as in Eq. (29), where u_d is assigned the task of absorbing (counteracting) the effects of the disturbance $w(t)$ in Eq. (46) and u_p is assigned the task of achieving the primary control objective $x_N(T_f)\approx 0$. Thus, Eq. (46) is written

$$\frac{d^2x_N}{dt^2}=u_p(t)+u_d(t)+w(t) \tag{47}$$

A. The Case of Constant Disturbances

To illustrate the design ideas while keeping the calculations relatively simple, we will restrict attention initially to the case where the uncertain disturbance waveform $w(t)$ is essentially *piecewise constant*. That is, over periods of time on the order of the final "u_d-absorber loop" settling time, the disturbance term $w(t)$ is assumed to change sufficiently slow that it may be approximated as an unknown constant. Thus, the uncertain disturbance $w(t)$ obeys the differential equation

$$\frac{dw(t)}{dt}=0 \qquad \text{almost everywhere} \tag{48}$$

which can be put in the form of Eq. (7) by setting $w(t)=z$ to obtain

$$w=z \qquad \dot{z}=\sigma(t) \qquad z=\text{state of } w(t) \tag{49}$$

where $\sigma(t)$ denotes a sparsely populated sequence of impulses with completely unknown arrival times and completely unknown intensities. Those unknown impulses are viewed as causing the once-in-a-while unknown jumps in the piecewise-constant disturbance $w(t)$.

The discrete-time models corresponding to Eqs. (47) and (49) are derived by first choosing the state variables $x_1=x_N$ and $x_2=\dot{x}_N$, and then employing Eqs. (9-15) to obtain

$$\begin{bmatrix} Ex_1(nT) \\ Ex_2(nT) \end{bmatrix}=\begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1(nT) \\ x_2(nT) \end{bmatrix}+\begin{bmatrix} T^2/2 \\ T \end{bmatrix}u_p(nT)+\begin{bmatrix} T^2/2 \\ T \end{bmatrix}u_d(nT)+\begin{bmatrix} T^2/2 \\ T \end{bmatrix}z(nT)+(\tilde{\gamma}) \tag{50a}$$

$$Ez(nT)=z(nT)+\tilde{\sigma} \tag{50b}$$

where T denotes the digital controller sampling period (not to be confused with the intercept time T_f). It is clear from Eq. (50a) that the complete absorbability condition Eq. (30) is satisfied for this example because $F\tilde{H}=\tilde{B}$. Thus, $\tilde{\Gamma}=1$ and therefore, following Eq. (31), the disturbance-absorber part u_d of the control can be designed as

$$u_d=-\hat{z}(nT) \tag{51}$$

where $\hat{z}(nT)$ is to be obtained from a state observer as described in Sec. IV. Using Eq. (51) in Eq. (50a), and recalling $\epsilon_z=z-\hat{z}$, the plant equation becomes

$$\begin{bmatrix} Ex_1(nT) \\ Ex_2(nT) \end{bmatrix}=\begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1(nT) \\ x_2(nT) \end{bmatrix}+\begin{bmatrix} T^2/2 \\ T \end{bmatrix}u_p(nT)+\begin{bmatrix} T^2/2 \\ T \end{bmatrix}z(nT)+(\tilde{\gamma}) \tag{52}$$

Now, since ϵ_z will presumably be designed to approach zero promptly, and since the residual $\tilde{\gamma}$ is completely unknown and sparse [see remarks following Eqs. (15) and (32)] the designer can proceed to design the remaining control term $u_p(nT)$ in Eq. (52) by considering the simplified model

$$\begin{bmatrix} Ex_1(nT) \\ Ex_2(nT) \end{bmatrix}=\begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1(nT) \\ x_2(nT) \end{bmatrix}+\begin{bmatrix} T^2/2 \\ T \end{bmatrix}u_p(nT) \tag{53}$$

The design of $u_p(nT)$ in Eq. (53) to achieve $x_1(T_f)\approx 0$ can now be accomplished by standard techniques; for instance, by introducing the discrete-time quadratic performance index

$$J=sx_1^2(T_f)+\sum_n\left[\langle x(nT),Qx(nT)\rangle+Ru_p^2(nT)\right]$$
$$(x=(x_1,x_2),s\gg 0,Q>0,R>0) \tag{54}$$

where $\langle\cdot,\cdot\rangle$ denotes inner product and employing well-known solution procedures for the discrete-time linear-quadratic regulator problem.[15] The final result is

$$\begin{aligned} u_p(nT) &= -[R+\tilde{b}^T K[(n+1)T]\tilde{b}]^{-1}[\tilde{b}^T K[(n+1)T]\tilde{A}]x(nT) \\ &= -\tilde{k}[(n+1)T]x(nT) \end{aligned} \tag{55}$$

where $K[(n+1)T]$ is a 2×2 symmetric matrix governed by a known matric Riccati difference equation; see Ref. 16; Eq. (25). Finally, the total DAC control $u=u_p+u_d$ is obtained from Eqs. (51) and (55) as

$$u(nT)=-\tilde{k}[(n+1)T]x(nT)-\hat{z}(nT) \tag{56}$$

If $x(nT)$ were not measurable directly, the control Eq. (56) would be implemented using $\hat{x}(nT)$ as obtained from a composite state observer.

The design of a state observer for implementation of the DAC control law Eq. (56) proceeds as follows. For simplicity, we assume that both of the plant state variables $x_1=x_N$, $x_2=\dot{x}_N$ can be measured directly; that is $C=I$ in Eq. (1) so that $y=(y_1,y_2)=(x_1,x_2)$. Consideration of cases where $C\neq I$, can be handled in a similar fashion, with a little more arithmetic. In principle, the designer can use either the full-dimensional composite state observer Eq. (18) or the reduced-dimension composite observer Eqs. (22) and (23) for any given problem. In the present example, since only $z(nT)$ remains to be estimated, we will elect to use the reduced-dimension observer Eqs. (22) and (23) and thereby avoid the unnecessary generation of plant state estimates $\hat{x}_1(nT)$, $\hat{x}_2(nT)$.

Since $C=I[=(2\times 2)]$, the two observer parameter matrices T_{12}, T_{22} in Eq. (21) can be chosen as $T_{12}=(0,0)^T$, $T_{22}=+1$, and therefore $\bar{T}_{12}=(0,0)$, $\bar{T}_{22}=+1$, and $C^{\#}=C=I$. The reduced-dimension (one-dimension) discrete-time observer is then given by the "digital filter" expression Eq. (22) and the assembly equation Eq. (23b) which, for this example, become the scalar expressions

$$\mathrm{E}\xi(nT)=\left(1+\Sigma_1\frac{T^2}{2}+\Sigma_2 T\right)\xi(nT)-\Sigma_1\left(\Sigma_1\frac{T^2}{2}+\Sigma_2 T\right)y_1(nT)$$

$$+\left[\Sigma_1 T-\Sigma_2\left(\Sigma_1\frac{T^2}{2}+\Sigma_2 T\right)\right]y_2(nT)+\left(\Sigma_1\frac{T^2}{2}+\Sigma_2 T\right)u(nT)$$

$$(y_1=x_1, y_2=x_2) \tag{57a}$$

and

$$\hat{z}(nT)=\xi(nT)-\Sigma_1 y_1(nT)-\Sigma_2 y_2(nT) \tag{57b}$$

The design of the observer gain matrix $\Sigma=(\Sigma_1,\Sigma_2)$ in Eqs. (57) is guided by the desired asymptotic stability behavior of the estimation errors Eq. (24). For the present example $(z-\hat{z})=\epsilon(nT)$ where, from Eq. (25), $\epsilon(nT)$ obeys the first-order scalar difference equation

$$\mathrm{E}\epsilon(nT)=\left(1+\Sigma_1\frac{T^2}{2}+\Sigma_2 T\right)\epsilon(nT) \tag{58}$$

Thus, if the designer desires the characteristic root (eigenvalue) $\tilde{\lambda}$ of the error difference equation Eq. (58) to be some a priori specified real value $\tilde{\lambda}=\tilde{\lambda}_0$, where $|\tilde{\lambda}_0|<1$, the gains (Σ_1,Σ_2) should be chosen to satisfy

$$\Sigma_2=\frac{\tilde{\lambda}_0-1}{T}-\frac{T}{2}\Sigma_1 \qquad |\tilde{\lambda}_0|<1 \tag{59}$$

The choice $\tilde{\lambda}_0=0$ is particularly attractive because then the estimation error $\epsilon(nT)$ in Eq. (58) theoretically reaches zero in one sample period (i.e., "deadbeat response" is realized).

This completeq the DAC design for this version of the example ($w(t)$ = piecewise constant). A block diagram of the continuous-time plant Eq. (46), with the discrete-time DAC controller Eqs. (56) and (57) installed, is shown in Fig. 6. This particular example of DAC design is comparable to the classical controller scheme known as "integral feedback." Note that the u_d loop reacts quickly to cancel the disturbance, even before $x(nT)\to 0$. Moreover, as $x(nT)\to 0$ in Fig. 6, the $u_d(nT)$ section of the controller becomes a T-unit "delayor" (1/E) with *unity* positive feedback and *zero input*. This means that the output of the delayor then "settles-down" to hold the constant value $w=z$ thereby continuing to cancel the constant external disturbance term $w(t)$. When the disturbance changes its "constant" value, the resulting perturbation in the plant output $y(t)$ drives the delayor loop to update the steady value of $u_d(nT)$.

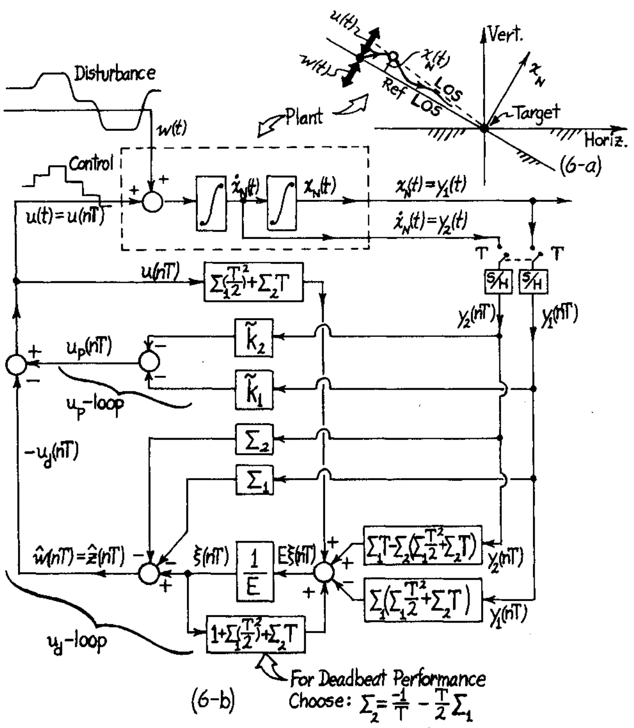

Fig. 6 Block diagram of plant and discrete-time DAC controller for homing missile example.

B. The Case of Exponential Disturbances

To demonstrate the critical-variable approach to disturbance minimization in this missile application we will now consider the case of uncertain "exponential" disturbances $w(t)$ in Eq. (46) having the form $w(t)=\bar{C}e^{\alpha t}$ where $\bar{C}$ is an unknown piecewise constant, and α is a known (real) constant. In this case, the disturbance differential equation Eq. (48) is replaced by $dw/dt=\alpha w$, almost everywhere, and Eqs. (49) and (50b) are replaced by

$$w=z \qquad \dot{z}=\alpha z+\sigma(t) \tag{60}$$

$$\mathrm{E}z(nT)=e^{\alpha T}z(nT)+\tilde{\sigma} \tag{61}$$

The matrix $\tilde{B}=(T^2/2,T)^T$ in Eq. (50) is unchanged by the introduction of the new disturbance model Eq. (60). However, the matrix $\widetilde{FH}$ in Eq. (50) *is* changed from $\widetilde{FH}=(T^2/2,T)^T$, where $(\ )^T$ denotes transpose, to the new expression

$$\widetilde{FH}=\left(\frac{e^{\alpha T}-1-\alpha T}{\alpha^2},\frac{e^{\alpha T}-1}{\alpha}\right)^T \tag{62}$$

Using Eq. (62) in Eq. (30) it is found that the complete absorbability condition for this example fails to be satisfied for $\alpha\neq 0$. This means physically that one cannot find a piecewise-constant control $u_d(nT)$ which simultaneously cancels the effect of an exponential disturbance $w(t)$ on *both* $x_N(nT)$ and $\dot{x}_N(nT)$. As indicated earlier, this limitation is not a shortcoming peculiar to DAC theory but rather is an inherent feature of *any* digital controller which uses piecewise-constant scalar control action $u(t)=u(nT)$ against a nonconstant disturbance.

Suppose the missile *position* variable $x_N(t)$ is judged to be the "critical variable" as far as accommodating disturbances is concerned. In other words, it is decided that the effect of $w(t)$ on $\dot{x}_N(nT)$ can be tolerated, so long as the effect of $w(t)$ on $x_N(nT)$ is completely cancelled (absorbed) by $u_d(nT)$. In that case, the designer can achieve disturbance absorption for the critical variable as follows. First, the equation governing the evolution of the critical variable $x_N(nT)=x_1(nT)$ must be derived. For this purpose, the existing expression for $\widetilde{FH}$ in Eq. (50a) is replaced by Eq. (62) and then one "operates" on $\mathrm{E}x_1(nT)$ in Eq. (50a) by the shift operator E( ) to obtain: [the residual terms $\tilde{\sigma}$, $\tilde{\gamma}$in Eq. (50) are hereafter disregarded since they do not affect the design procedure]

$$\mathrm{E}^2x_1(nT)=\mathrm{E}x_1(nT)+T\mathrm{E}x_2(nT)+\frac{T^2}{2}\mathrm{E}u_p(nT)$$

$$+\frac{T^2}{2}\mathrm{E}u_d(nT)+\left(\frac{e^{\alpha T}-1-\alpha T}{\alpha^2}\right)\mathrm{E}z(nT) \tag{63}$$

However, the term $T\mathrm{E}x_2(nT)$ in Eq. (63) can be written, using Eq. (50a) as

$$T\mathrm{E}x_2 = \mathrm{E}x_1(nT) - x_1(nT) + \frac{T^2}{2}u_p(nT) + \frac{T^2}{2}u_d(nT) + \left[\frac{T\alpha(e^{\alpha T}-1) - e^{\alpha T} + 1 + \alpha T}{\alpha^2}\right]z(nT) \tag{64}$$

Incorporating Eqs. (61) and (64) in Eq. (63) the desired difference equation governing $x_1(nT)$ is finally obtained as

$$\mathrm{E}^2x_1(nT) = 2\mathrm{E}x_1(nT) - x_1(nT) + \frac{T^2}{2}[u_p(nT) + \mathrm{E}u_p(nT)] + \frac{T^2}{2}[u_d(nT) + \mathrm{E}u_d(nT)] + \left[\frac{e^{\alpha T}-1}{\alpha}\right]^2 z(nT) \tag{65}$$

It is now clear from Eq. (65) that the total effect of $w(t)$ on the "critical variable" $x_1(nT)$ can be *completely absorbed* if, and only if, one designs $u_d(nT)$ to satisfy

$$\frac{T^2}{2}[u_d(nT) + \mathrm{E}u_d(nT)] = -\left[\frac{e^{\alpha T}-1}{\alpha}\right]^2 z(nT) \tag{66}$$

Condition Eq. (66) is recognized as a first-order difference equation for generation of the required control $u_d(nT)$. In other words, in this example the u_d part of the DAC controller turns out to be a *dynamic compensator* Eq. (66) even *before* one introduces the dynamic state observer to generate $\hat{z}(nT)$. Thus, to physically implement Eq. (66) one must synthesize the recursive digital filter

$$\mathrm{E}u_d(nT) = -u_d(nT) - \frac{2}{T^2}\left[\frac{e^{\alpha T}-1}{\alpha}\right]^2 \hat{z}(nT) \tag{67}$$

to produce on-line, real-time values of $u_d(nT)$ from the "input" data $\hat{z}(nT)$, where $\hat{z}(nT)$ is generated by the state-observer digital filter Eq. (57). Note that the output $u_d(nT)$ of Eq. (67) must be summed with $u_p(nT)$ and fed back into the state observer Eq. (57).

The dynamic compensator Eq. (67) will ideally absorb all effects of $w(t)$ on $x_N(nT)$ but will not necessarily reduce the effects of $w(t)$ on $\dot{x}_N(nT)$. Thus, in practice a performance trade-off study between the norm-minimization strategy Eqs. (33-35) and the critical-variable strategy Eq. (67) might be useful in deciding which strategy is "best" for each specific problem. Of course, there are many other candidate approaches to disturbance minimization which might be included in such a design competition (see Ref. 5, pp. 455-468).

Assuming that $u_d(nT)$ is designed in accordance with Eq. (67), the remaining primary control term $u_p(nT)$ in Eq. (65) can be designed to achieve $x_1(T_f) \approx 0$ by disregarding the last two terms on the right side of Eq. (65) and employing the well-known linear quadratic optimal control method[16] as illustrated in Eqs. (54) and (55). For this latter purpose, it should be noted that the state equation chosen for Eq. (65) will probably differ from Eq. (53) because Eq. (65) involves the input "derivative" term $\mathrm{E}u_p(nT)$ and therefore the choice of the second state variable x_2 for Eq. (65) will probably differ from that chosen in Eq. (53) (see Ref. 20, p. 697).

VII. Conclusions

The discrete-time version of disturbance-accommodating control theory developed here can be used as a design tool for synthesizing high-performance digital control systems. Such controllers have a reserve capability for adapting to, and coping with, a wide variety of realistic, uncertain disturbances. Missile control systems utilizing digital controllers designed by the procedures described herein should exhibit an exceptionally high degree of guidance accuracy in the face of wind gusts, target maneuvers, and other uncertain disturbances.

This paper has considered the disturbance-absorption and disturbance-minimization modes of accommodation. In a subsequent paper, the design of discrete-time *disturbance-utilizing* DAC controllers will be described.

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